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ON DIAMETERS OF ORBITS OF COMPACT GROUPS IN UNITARY REPRESENTATIONS

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Abstract

For a compact group G, we compute the Kazhdan constants $\kappa(G, G)$ obtained by taking G itself as a generating subset. We get $\kappa(G, G) = \sqrt{2n/(n-1)}$ if G is finite of order n, and $\kappa(G, G) = \sqrt{2}$ if G is infinite.

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1. Introduction

Let H be a locally compact group, and let π be a strongly continuous, unitary representation of H on a Hilbert space $\mathscr{H}_\pi$. To understand the metric structure of orbits of $\pi(H)$ on $\mathscr{H}_\pi$, we may clearly, by linearity, restrict to orbits in the unit sphere $\mathscr{H}_\pi^1$. Plainly, the diameter of such an orbit is at most 2. It follows from Corollary 11 of Chapter 3 in [7] that the following statements are equivalent:

(i) π has a non-zero fixed vector;

(ii) there exists an orbit of $\pi(H)$ in $\mathscr{H}_\pi^1$ with diameter $< \sqrt{2}$.

(the non-trivial implication (ii) implies (i) is proved by considering the centre of the unique smallest ball in $\mathscr{H}_\pi$ containing a given orbit in $\mathscr{H}_\pi^1$; this is clearly a fixed vector for $\pi(H)$ and, if the diameter of the orbit is $< \sqrt{2}$, this fixed vector is non-zero).

In this paper, we deal with (strongly continuous, unitary) representations of a compact group G. Our first result (Proposition 1 in Section 2) states that in this case the above characterization of representations with non-zero fixed vectors remains true even if we allow orbits of diameter $\sqrt{2}$ in $\mathscr{H}_\pi^1$.

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Now, assume that G is a finite group of order n. Our second result, proved in Section 3, says that in this case the constant $\sqrt{2}$ of the preceding result can be increased, namely:

PROPOSITION 2. *Let G be a finite group of order n. Let π be a unitary representation of G without non-zero fixed vector. Then every orbit in $\mathscr{H}_\pi^1$ has diameter at least $\sqrt{2n/(n-1)}$.*

Our main result, proved in Section 4, is that the bounds $\sqrt{2n/(n-1)}$ for finite groups of order n, and $\sqrt{2}$ for infinite compact groups, are sharp. More precisely, for G a compact group, denote by λ_0 the restriction of the left regular representation λ of G to the orthogonal complement of constant functions in $L^2(G)$. Then we have:

THEOREM 1. *For G a finite group of order n, the minimum of the diameters of orbits in $\mathscr{H}_{\lambda_0}^1$ is $\sqrt{2n/(n-1)}$.*

For G an infinite compact group, the infimum of the diameters of orbits in $\mathscr{H}_{\lambda_0}$ is $\sqrt{2}$.

Notice that, in the second case, the infimum is not attained because of Proposition 1.

Our results may be interpreted in terms of Kazhdan constants, which we now define. From now on representations are always assumed to be strongly continuous and unitary. For H a locally compact group, K a compact subset of H, and π a representation of H, we define the Kazhdan constant $\kappa(\pi, H, K)$ by:

$$\kappa(\pi, H, K) = \inf_{\xi \in \mathscr{H}_\pi^1} \max_{k \in K} \|\pi(k)\xi - \xi\|.$$

We also define

$$\kappa(H, K) = \inf \kappa(\pi, H, K),$$

where the infimum is taken over all representations π of H with no non-zero invariant vector. These definitions can be found in [6]; see also the problem mentioned in [7, Chapter 1, no 17]. For explicit computations of Kazhdan constants, see [1, 2, 3].

The first result mentioned above may be rephrased by saying that, if G is a compact group and π a representation of G with no non-zero fixed vector, then $\kappa(\pi, G, G) \geq \sqrt{2}$. Proposition 2 and Theorem 1 above then become:

PROPOSITION 2$'$. *If G is a finite group of order n, then for any representation π of G without non-zero fixed vector, $\kappa(\pi, G, G) = \sqrt{2n/(n-1)}$.*

THEOREM 1$'$. *If G is a finite group of order n, then $\kappa(\lambda_0, G, G) = \kappa(G, G) = \sqrt{2n/(n-1)}$. If G is an infinite compact group, then $\kappa(\lambda_0, G, G) = \kappa(G, G) = \sqrt{2}$.*

For finite cyclic groups (respectively, for the circle group) the above results were obtained by the first author in [4] (respectively, [5]).

2. Orbits of diameter $\sqrt{2}$

PROPOSITION 1. *Let G be a compact group, and π be a representation of G. The following are equivalent:*

(i) *π has a non-zero fixed vector;*
(ii) *there exists in $\mathcal{H}_\pi^1$ an orbit of $\pi(G)$ with diameter $\leq \sqrt{2}$.*

PROOF. In view of the result stated in the beginning of Section 1, it is enough to show that, if there exists some vector ξ in $\mathcal{H}_\pi^1$ with diameter $\sqrt{2}$, then π has non-zero fixed vectors. But from $\max_{g\in G} \|\pi(g)\xi - \xi\|^2 = 2$, we deduce $\min_{g\in G} \mathrm{Re}\langle\pi(g)\xi|\xi\rangle = 0$. The function $G \to \mathbb{R}$, defined by $g \mapsto \mathrm{Re}\langle\pi(g)\xi|\xi\rangle$ is continuous, non-negative, and equal to 1 at the identity of G. So, denoting by dg the normalized Haar measure on the compact group G, one has $\mathrm{Re}\left[\int_G \langle\pi(g)\xi|\xi\rangle dg\right] > 0$. Therefore the vector $\int_G \pi(g)\xi\, dg$ is a non-zero fixed vector for π.

REMARK. Proposition 1 holds only for compact groups. Indeed, for a non-compact locally compact group G, the left regular representation λ of H on $L^2(H)$ clearly has no non-zero invariant vector, but it has orbits of diameter $\sqrt{2}$ in $L^2(H)^1$. To see this, let us fix a compact subset K with non-empty interior in H; we denote by χ_K the characteristic function of K, and by $\mu(K) > 0$ its Haar measure; then the function $\mu(K)^{-1/2}\chi_K$ is a unit vector in $L^2(H)$, whose orbit under $\lambda(H)$ has diameter $\sqrt{2}$.

3. The case of finite groups

We now prove Proposition 2. Let G be a finite group of order n. Let π be a representation of G with no non-zero fixed vector. We have to show that, for any $\xi \in \mathcal{H}_\pi^1$, the orbit ξ has diameter at least $\sqrt{2n/(n-1)}$.

Since π has no non-zero invariant vector, we have $\sum_{g\in G} \pi(g) = 0$. Thus

$$\frac{1}{n-1} \sum_{g\in G-\{e\}} \langle\pi(g)\xi|\xi\rangle = \frac{-1}{n-1},$$

which implies

$$\min_{g\in G} \mathrm{Re}\langle\pi(g)\xi|\xi\rangle \leq \frac{-1}{n-1}.$$

Then:

$$\max_{g\in G} \|\pi(g)\xi - \xi\|^2 = 2 - 2\min_{g\in G} \mathrm{Re}\langle\pi(g)\xi|\xi\rangle \geq 2 + \frac{2}{n-1} = \frac{2n}{n-1}.$$

4. The main result

We present here a more general result, from which Theorem 1 will follow easily.

THEOREM 2. *Let $(X, \mathscr{B}, \mu)$ be a probability space. Define a constant $m(X)$ by*

$$m(X) = \inf\{\mu(B) : B \in \mathscr{B}, \mu(B) > 0\}.$$

Let H be a locally compact group acting measurably on X by measure-preserving automorphisms. Denote by σ the corresponding representation on $L^2(X)$, and by σ_0 the restriction of σ to $L^2_0(X)$, the orthogonal complement of constant functions in $L^2(X)$. Then, for any compact subset K of H, $\kappa(\sigma_0, H, K) \leq \sqrt{2/(1-m(X))}$.

Theorem 1 is then deduced as follows. We let the compact group G act on itself by left translations, and distinguish two cases.

(a) G is discrete, that is, G is finite, say of order n. Then $m(G) = 1/n$ and $\kappa(\lambda_0, G, G) \leq \sqrt{2n/(n-1)}$;
(b) G is not discrete, that is, G is infinite. Then $m(G) = 0$ and $\kappa(\lambda_0, G, G) \leq \sqrt{2}$.

PROOF OF THEOREM 2. Let P be the orthogonal projection from $L^2(X)$ onto $L^2_0(X)$; for $\xi \in L^2(X)$, we have $P\xi = \xi - \langle \xi | 1 \rangle$. Let us fix $B \in \mathscr{B}$ such that $0 < \mu(B) < 1$, and define a function ξ_B by:

$$\xi_B(x) = \begin{cases} \mu(B)^{-1/2} & \text{if } x \in B, \\ 0 & \text{otherwise.} \end{cases}$$

So $\xi_B \in \mathscr{H}^1_\sigma$. Let then K be any compact subset of H; one has:

$$\begin{aligned} \max_{h \in K} \left\| \sigma_0(h) \frac{P\xi_B}{\|P\xi_B\|} - \frac{P\xi_B}{\|p\xi_B\|} \right\|^2 &= \max_{h\in K} \frac{1}{\|P\xi_B\|^2} \|\sigma(h)\xi_B - \xi_B\|^2 \\ = \max_{h \in K} \frac{2(1 - \langle \sigma(h)\xi_B | \xi_B \rangle)}{1 - \langle \xi_B | 1 \rangle^2} &= \max_{h \in K} \frac{2}{1-\mu(B)} \left(1 - \frac{\mu(hB \cap B)}{\mu(B)}\right) \\ \leq \frac{2}{1-\mu(B)}. \end{aligned}$$

The result then follows from the definitions.

REMARK. Let G be a finite group of order n. Then $\lambda_0(G)$ admits in $\mathscr{H}^1_{\lambda_0}$ an orbit of cardinality n such that the distance between any two distinct points of the orbit is $\sqrt{2n/(n-1)}$ (in other words, the convex hull of this orbit is a regular simplex of dimension $n-1$). To see this, let us start with the n characteristic functions of the elements of G; this is an orbit of $\lambda(G)$ in $\mathscr{H}^1_\lambda$, and any two distinct points in this orbit are $\sqrt{2}$ apart. We then project this orbit orthogonally onto $\ell_0^2(G)$ and, as in the proof of Theorem 2, re-scale it to make its vectors of norm 1; so we obtain the desired orbit in $\lambda_0(G)$.

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